

VARIETIES OF MINIMAL RATIONAL TANGENTS ON DOUBLE COVERS OF PROJECTIVE SPACE

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ABSTRACT. Let $\phi : X \rightarrow \mathbb{P}^n$ be a double cover branched along a smooth hypersurface of degree $2m, 2 \leq m \leq n - 1$. We study the varieties of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at a general point x of X . We describe the homogeneous ideal of \mathcal{C}_x and show that the projective isomorphism type of \mathcal{C}_x varies in a maximal way as x varies over general points of X . Our description of the ideal of \mathcal{C}_x implies a certain rigidity property of the covering morphism ϕ . As an application of this rigidity, we show that any finite morphism between such double covers with $m = n - 1$ must be an isomorphism. We also prove that Liouville-type extension property holds with respect to minimal rational curves on X .

KEYWORDS. double covers of projective space, Fano manifolds, varieties of minimal rational tangents

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1. INTRODUCTION

Throughout the paper, we will work over the field of complex numbers. Let X be a Fano manifold of Picard number 1. For a general point $x \in X$, a rational curve through x is called a minimal rational curve if its degree with respect to K_X^{-1} is minimal among all rational curves through x . Denote by \mathcal{K}_x the space of minimal rational curves through x . The projective subvariety $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ defined as the union of tangent directions to members of \mathcal{K}_x is called the variety of minimal rational tangents (VMRT) at x . The projective geometry of \mathcal{C}_x plays a key role in understanding the geometry of X , often leading to a certain rigidity phenomenon (cf. the survey [3]) on X . This motivates the study of the geometry of $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ for various examples of X .

In the current article, we study the case when X is a double cover $\phi : X \rightarrow \mathbb{P}^n, n \geq 3$, of projective space \mathbb{P}^n branched along a smooth hypersurface $Y \subset \mathbb{P}^n$ of degree $2m, 2 \leq m \leq n - 1$. Although this is one of the basic examples of Fano manifolds, its VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ has not been described explicitly. Our first result is the following description of the defining equations of the VMRT.

Theorem 1.1. *For a double cover $X \rightarrow \mathbb{P}^n, n \geq 3$, branched along a smooth hypersurface of degree $2m, 2 \leq m \leq n - 1$, the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at a general point $x \in X$ is a smooth complete intersection of multi-degree $(m + 1, m + 2, \dots, 2m)$.*

It is enlightening to compare Theorem 1.1 with the case when X is a smooth hypersurface of degree $m, 2 \leq m \leq n$, in \mathbb{P}^{n+1} . In the latter case, it is classical that the VMRT at a general point is a smooth complete intersection of multi-degree $(2, 3, \dots, m)$ (e.g. Example 1.4.2 in [3] or [9]).

In the course of proving Theorem 1.1, we will also prove the following partial converse to it.

Theorem 1.2. *Let $Z \subset \mathbb{P}^{n-1}$, $n \geq 3$, be a general complete intersection of multi-degree $(m+1, m+2, \dots, 2m)$ with $2 \leq m \leq n-1$. Then there exists a smooth hypersurface $Y \subset \mathbb{P}^n$ of degree $2m$ such that a double cover X of \mathbb{P}^n branched along Y has a point $x \in X$ with its VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ isomorphic to $Z \subset \mathbb{P}^{n-1}$.*

Theorem 1.1 and Theorem 1.2 are proved by explicit computation for a certain choice of Y , based on the fact that minimal rational curves of X correspond to lines of \mathbb{P}^n which have even contact order with $Y \subset \mathbb{P}^n$ as recalled in Proposition 2.4. Then Theorem 1.1 for arbitrary smooth Y can be obtained by a flatness argument.

Our explicit computation enables us to study also the variation of the VMRT \mathcal{C}_x as x varies over X . Describing the variation of VMRT is not an easy problem even for very simple Fano manifolds, such as hypersurfaces in \mathbb{P}^{n+1} . In [9], Landsberg and Robles proved that when X is a general hypersurface of degree $\leq n$ in \mathbb{P}^{n+1} , the VMRT at general points of X have maximal variation. We will prove the following analogue of their result in our setting.

Theorem 1.3. *Let $Y \subset \mathbb{P}^n$, $n \geq 4$, be a general hypersurface of degree $2m$, $2 \leq m \leq n-1$, and let X be a double cover of \mathbb{P}^n branched along Y . Then the family of VMRT's*

$$\{\mathcal{C}_x \subset \mathbb{P}T_x(X) \mid \text{general } x \in X\}$$

has maximal variation. More precisely, for a general point $x \in X$, choose a trivialization of $\mathbb{P}T(U) \cong \mathbb{P}^{n-1} \times U$ in a neighborhood U of x . Define a morphism $\zeta : U \rightarrow \text{Hilb}(\mathbb{P}^{n-1})$ by $\zeta(y) := [\mathcal{C}_y]$ for $y \in U$. Then the rank of $d\zeta_x$ is n and the intersection of the image of ζ and the $GL(n, \mathbb{C})$ -orbit of $\zeta(x)$ is isolated at $\zeta(x)$.

The condition $n \geq 4$ in Theorem 1.3 excludes the case of $(n, m) = (3, 2)$. It is likely that the statement of Theorem 1.3 holds also for this case. However, this case seems to require much more complicated computation.

As mentioned at the beginning, the geometry of \mathcal{C}_x often leads to a certain rigidity result. What rigidity phenomenon does our description of \mathcal{C}_x exhibit? The double covering morphism $\phi : X \rightarrow \mathbb{P}^n$ sends members of \mathcal{K}_x to lines. We show that this property characterizes ϕ in the following strong sense.

Theorem 1.4. *Let $Y \subset \mathbb{P}^n$, $n \geq 3$, be a smooth hypersurface of degree $2m$, $2 \leq m \leq n-1$, and let $\phi : X \rightarrow \mathbb{P}^n$ be a double cover branched along Y . Let $U \subset X$ be a neighborhood (in classical topology) of a general point $x \in X$ and $\varphi : U \rightarrow \mathbb{P}^n$ be a biholomorphic immersion such that for any member C of \mathcal{K}_y , $y \in U$, the image $\varphi(C \cap U)$ is contained in a line in \mathbb{P}^n . Then there exists a projective transformation $\psi : \mathbb{P}^n \rightarrow \mathbb{P}^n$ such that $\varphi = \psi \circ (\phi|_U)$.*

As a consequence, we obtain the following algebraic version.

Corollary 1.5. *In the setting of Theorem 1.4, let \hat{X} be an n -dimensional projective variety equipped with generically finite surjective morphisms $g : \hat{X} \rightarrow X$ and $h : \hat{X} \rightarrow \mathbb{P}^n$ such that for a minimal rational curve C through a general point of X , there exists an irreducible component C' of $g^{-1}(C)$ whose image $h(C') \subset \mathbb{P}^n$ is a line. Then there exists an automorphism $\psi : \mathbb{P}^n \rightarrow \mathbb{P}^n$ such that $h = \psi \circ \phi \circ g$.*

This is a remarkable property of the double cover $\phi : X \rightarrow \mathbb{P}^n$, because an analogous statement fails drastically for many examples of Fano manifolds of Picard number 1 as the following example shows.

Example 1.6. Let $X \subset \mathbb{P}^N$ be a Fano manifold embedded in projective space of dimension

X are lines of \mathbb{P}^N lying on X . There are many such examples, e.g., rational homogeneous spaces under a minimal embedding or complete intersections of low degree in \mathbb{P}^N . We can define a finite projection $\phi : X \rightarrow \mathbb{P}^n$ to a linear subspace $\mathbb{P}^n \subset \mathbb{P}^N$ by choosing a suitable linear subspace $\mathbb{P}^{N-n-1} \subset (\mathbb{P}^N \setminus X)$. Then minimal rational curves are sent to lines in \mathbb{P}^n by ϕ . There are many different ways to choose such \mathbb{P}^{N-n-1} and projections. For most examples of X , different choices of ϕ need not be related by projective transformations of \mathbb{P}^n .

What makes the difference between Corollary 1.5 and Example 1.6? The key point is that the ideal defining the VMRT of the double cover, as described in Theorem 1.1, does not contain a quadratic polynomial. In fact, we will prove a general version, Theorem 5.4, of Theorem 1.4 where the double cover X is replaced by any Fano manifold whose VMRT at a general point is not contained in a hyperquadric. In this regard, we should mention that our double cover $\phi : X \rightarrow \mathbb{P}^n$ is the *first known example* of a Fano manifold with Picard number 1 whose VMRT at a general point is not contained in a hyperquadric. Note that the VMRT's of Fano manifolds in Example 1.6 are contained in hyperquadrics coming from the second fundamental form of $X \subset \mathbb{P}^N$.

Theorem 1.4 has an application in the study of morphisms between double covers. There have been several works, e.g., [1], [2], [6], [7] and [11], classifying finite morphisms between Fano threefolds of Picard number 1. But there still remain a few unsettled cases. One such case is finite morphisms between double covers of \mathbb{P}^3 branched along smooth quartic surfaces. When the quartic surfaces do not contain lines, Theorem 1.5 (more precisely, in the subsection (4.2.2)) of [11] proves that such a morphism must be an isomorphism. However, the approach of [11] has technical difficulties when the quartic surfaces contain lines. Using Theorem 1.4, we can settle this case. More precisely, we obtain the following.

Theorem 1.7. *Let $Y_1, Y_2 \subset \mathbb{P}^n, n \geq 3$, be two smooth hypersurfaces of degree $2n - 2$. Let $\phi_1 : X_1 \rightarrow \mathbb{P}^n$ (resp. $\phi_2 : X_2 \rightarrow \mathbb{P}^n$) be a double cover of \mathbb{P}^n branched along Y_1 (resp. Y_2). Suppose there exists a finite morphism $f : X_1 \rightarrow X_2$. Then f is an isomorphism.*

Another application of Theorem 1.4 is the following problem, which is Problem 7.9 in [4].

Problem 1.8 (Liouville-type extension problem). Let X be a Fano manifold of Picard number 1. Let U_1 and U_2 be two connected open subsets (in classical topology) in X . Suppose that we are given a biholomorphic map $\gamma : U_1 \rightarrow U_2$ such that for any minimal rational curve $C \subset X$, there exists another minimal rational curve C' with $\gamma(U_1 \cap C) = U_2 \cap C'$. Then does there exist $\Gamma \in \text{Aut}(X)$ with $\Gamma|_{U_1} = \gamma$?

Problem 1.8 is called Liouville-type extension, because Liouville's theorem in conformal geometry gives an affirmative answer to Problem 1.8 when X is a smooth quadric hypersurface in \mathbb{P}^{n+1} . An affirmative answer to Problem 1.8 is known if $\dim \mathcal{K}_x > 0$ for a general $x \in X$ (this is essentially proved in [5]). However, when $\dim \mathcal{K}_x = 0$, affirmative answers are known only in a small number of examples of X , such as hypersurfaces of degree n in \mathbb{P}^{n+1} and Mukai-Umemura threefolds (see Section 7 of [4]).

Theorem 1.4 enables us to give a stronger form of Liouville-type extension for our double cover X as follows.

Theorem 1.9. *Let $Y_1, Y_2 \subset \mathbb{P}^n, n \geq 3$, be two smooth hypersurfaces of degree $2m, 2 \leq m \leq n - 1$. Let $\phi_1 : X_1 \rightarrow \mathbb{P}^n$ (resp. $\phi_2 : X_2 \rightarrow \mathbb{P}^n$) be a double cover of \mathbb{P}^n branched along Y_1 (resp. Y_2). Let $U_1 \subset X_1$ and $U_2 \subset X_2$ be two connected open subsets (in classical topology) such that for any minimal rational curve $C \subset X_1$, there exists another minimal rational curve $C' \subset X_2$ with $\phi_1(U_1 \cap C) = \phi_2(U_2 \cap C')$. Then does there exist $\Gamma \in \text{Aut}(X_1)$ with $\Gamma|_{U_1} = \gamma$?*

given a biholomorphic map $\gamma : U_1 \rightarrow U_2$ such that for any minimal rational curve $C_1 \subset X_1$, there exists a minimal rational curve $C_2 \subset X_2$ with $\gamma(U_1 \cap C_1) = U_2 \cap C_2$. Then we can find a biregular morphism $\Gamma : X_1 \rightarrow X_2$ with $\Gamma|_{U_1} = \gamma$.

The organization of this paper is as follows. In Section 2, we will present some basic facts concerning double covers of \mathbb{P}^n and their minimal rational curves. Theorem 1.1 and Theorem 1.2 will be proved in Section 3. In Section 4, the variation of VMRT is studied and Theorem 1.3 will be proved. Finally, in Section 5, we review the notion of projective connections to prove a general version of Theorem 1.4 and explain how Theorem 1.7 and Theorem 1.9 can be derived from Theorem 1.4.

2. MINIMAL RATIONAL CURVES AND ECO LINES

Throughout, we will fix integers $n \geq 3$ and $2 \leq m \leq n - 1$. Let $Y \subset \mathbb{P}^n$ be a smooth hypersurface of degree $2m$. Let $\phi : X \rightarrow \mathbb{P}^n$ be a double cover of \mathbb{P}^n ramified along Y . Such a double cover arises as a submanifold in the line bundle $\mathcal{O}_{\mathbb{P}^n}(m)$ as explained in pp.242-244 of [10]. This implies the following uniqueness result, where the slightly awkward appearance of the open subsets U_1 and U_2 are for our later use in Section 5.

Lemma 2.1. *Given a smooth hypersurface $Y \subset \mathbb{P}^n$ of degree $2m$, let $\phi_1 : X_1 \rightarrow \mathbb{P}^n$ and $\phi_2 : X_2 \rightarrow \mathbb{P}^n$ be two choices of double covers of \mathbb{P}^n branched along Y . Let $U_1 \subset X_1$ and $U_2 \subset X_2$ be two connected open subsets (in classical topology) with $\phi_1(U_1) = \phi_2(U_2)$. Then there exists a biregular morphism $\Gamma : X_1 \rightarrow X_2$ with $\Gamma(U_1) = U_2$ and $\phi_1 = \phi_2 \circ \Gamma$.*

Definition 2.2. Let $\phi : X \rightarrow \mathbb{P}^n$ be a double cover branched along a smooth hypersurface $Y \subset \mathbb{P}^n$ of degree $2m$. A rational curve $C \subset X$ with $\phi(C) \not\subset Y$ is a *minimal rational curve* if it has degree 1 with respect to $\phi^*\mathcal{O}_{\mathbb{P}^n}(1)$. For $x \in X \setminus \phi^{-1}(Y)$, we denote by \mathcal{K}_x the (normalized) space of minimal rational curves through x . It is known (e.g. II.3.11.5 in [8]) that \mathcal{K}_x is a union of finitely many nonsingular projective varieties. From the adjunction formula

$$K_X = \phi^*(K_{\mathbb{P}^n} + \frac{1}{2}[Y]) = \phi^*\mathcal{O}_{\mathbb{P}^n}(-n + m - 1),$$

X is a Fano manifold of index $n - m + 1$ and $\dim \mathcal{K}_x = n - m - 1$.

Definition 2.3. Let $Y \subset \mathbb{P}^n$ be an irreducible reduced hypersurface. A line $\ell \subset \mathbb{P}^n$ is an *ECO (Even Contact Order) line* with respect to Y if $\ell \not\subset Y$ and the local intersection number at each point of $\ell \cap Y$ is even. For a point $x \in \mathbb{P}^n \setminus Y$, identify the space of lines through x with the projective space $\mathbb{P}T_x(\mathbb{P}^n)$ and denote by $\mathcal{E}_x^Y \subset \mathbb{P}T_x(\mathbb{P}^n)$ the space of ECO lines through x with respect to Y .

Next proposition is a direct generalization of well-known facts for $(n, m) = (3, 2)$ (e.g. [12]).

Proposition 2.4. *In the setting of Definition 2.2, an irreducible reduced curve $C \subset X$ with $\phi(C) \not\subset Y$ is a minimal rational curve if and only if the image curve $\phi(C) \subset \mathbb{P}^n$ is an ECO line with respect to Y . Moreover, a minimal rational curve C is smooth and $\phi|_C : C \rightarrow \phi(C)$ is an isomorphism.*

Proof. Let $C \subset X$ be an irreducible curve such that $\ell := \phi(C)$ is an ECO line with respect to Y . Suppose that $\phi_C : C \rightarrow \ell$ is not birational, i.e., $C = \phi^{-1}(\phi(C))$. For a point $z \in \phi(C) \cap Y$, let t be a local uniformizing parameter on ℓ at z and let r_z be the local intersection number of ℓ and Y at z . Then C is analytically defined by the equation $s^2 = t^{r_z}$ (cf. [12], p.242-244). Let \tilde{C} be the normalization of C . Then \tilde{C} is a smooth curve and $\phi|_{\tilde{C}} : \tilde{C} \rightarrow \ell$ is a birational map.

$z \in \phi(C) \cap Y$, the composition of the normalization morphism $\tilde{C} \rightarrow C$ and the covering morphism $\phi|_C : C \rightarrow \ell$ induces a morphism $\tilde{C} \rightarrow \ell$ of degree 2 without ramification point, a contradiction. Thus $\phi|_C : C \rightarrow \ell$ is birational and C has degree 1 with respect to $\phi^* \mathcal{O}_{\mathbb{P}^n}(1)$.

Conversely, if C is a minimal rational curve, then $\ell := \phi(C)$ is a line in \mathbb{P}^n with $\ell \not\subset Y$ and $\phi|_C : C \rightarrow \ell$ must be birational. Thus $\phi^{-1}(\ell)$ has an irreducible component C' different from C with $\phi(C \cap C') = \ell \cap Y$. By the same argument as before, if the local intersection number r_z at $z \in \ell \cap Y$ is odd, the germ of $\phi^{-1}(\ell)$ over z , defined by $s^2 = t^{r_z}$, is irreducible, a contradiction. Thus r_z is even for all $z \in \ell \cap Y$ and ℓ is an ECO line. Moreover, C must be smooth and the morphism $\phi|_C : C \rightarrow \ell$ is an isomorphism. \square

We have the following consequence.

Proposition 2.5. *In the setting of Proposition 2.4, let $Y' \subset \mathbb{P}^n$ be an irreducible reduced hypersurface distinct from Y . Then a general ECO line with respect to Y intersects Y' transversally. In particular, a general ECO line with respect to Y cannot be an ECO line with respect to Y' .*

Proof. On a Fano manifold X , for any subset $Z \subset X$ of codimension ≥ 2 and any reduced hypersurface $D \subset X$, a general minimal rational curve is disjoint from Z (e.g. Lemma 2.1 in [3]) and intersects D transversally (the proof is similar to the proof of Lemma 2.1 in [3]). Putting $Z = \phi^{-1}(Y \cap Y')$ and $D = \phi^{-1}(Y')$ for our $\phi : X \rightarrow \mathbb{P}^n$ branched along Y , we see that a general minimal rational curve C intersects $\phi^{-1}(Y')$ transversally and $\phi(C) \cap Y \cap Y' = \emptyset$. Thus $\phi(C)$ intersects Y' transversally. \square

Proposition 2.6. *In the setting of Proposition 2.4, let x be a general point of X . Let $\tau_x : \mathcal{K}_x \rightarrow \mathbb{P}T_x(X)$ be the tangent morphism associating each member of \mathcal{K}_x its tangent direction at x . Then τ_x is an embedding and the VMRT $\mathcal{C}_x = \text{Im}(\tau_x) \subset \mathbb{P}T_x(X)$ is a nonsingular projective variety with finitely many components of dimension $n - m - 1$, isomorphic to $\mathcal{E}_{\phi(x)} \subset \mathbb{P}T_{\phi(x)}(\mathbb{P}^n)$.*

Proof. The differential $d\phi_x : \mathbb{P}T_x(X) \rightarrow \mathbb{P}T_{\phi(x)}(\mathbb{P}^n)$ sends $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ isomorphically to $\mathcal{E}_{\phi(x)}^Y \subset \mathbb{P}T_{\phi(x)}(\mathbb{P}^n)$ by Proposition 2.4. It follows that τ_x is injective because lines on \mathbb{P}^n are determined by their tangent directions.

Since we know that \mathcal{K}_x is nonsingular of dimension $n - m - 1$, to prove that τ_x is an embedding, it remains to show that τ_x is an immersion. By Proposition 1.4 in [3], this is equivalent to showing that for any member $C \subset X$ of \mathcal{K}_x , the normal bundle $N_{C/X}$ satisfies

$$N_{C/X} = \mathcal{O}_{\mathbb{P}^1}(1)^{n-m-1} \oplus \mathcal{O}_{\mathbb{P}^1}^m.$$

By the generality of x , we can write

$$N_{C/X} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_{n-1})$$

for integers $a_1 \geq \cdots \geq a_{n-1} \geq 0$ satisfying $\sum_i a_i = n - m - 1$. Since ϕ is unramified at general points of C and $\phi|_C : C \rightarrow \ell := \phi(C)$ is an isomorphism, we have an injective sheaf homomorphism

$$\phi_* : N_{C/X} \rightarrow N_{\ell/\mathbb{P}^n} = \mathcal{O}(1)^{n-1}.$$

Thus $a_1 \leq 1$. It follows that $a_1 = \cdots = a_{n-m-1} = 1$ and $a_{n-m} = \cdots = a_{n-1} = 0$. \square

3. DEFINING EQUATIONS OF VMRT

Definition 3.1. A polynomial $A(t_1, \dots, t_m)$ in m variables is said to be *weighted homogeneous of weighted degree k* if it is of the form

$$A(t_1, \dots, t_m) = \sum c_{i_1, \dots, i_m} t_1^{i_1} \cdots t_m^{i_m}$$

with coefficients $c_{i_1, \dots, i_m} \in \mathbb{C}$. An equivalent way of defining it is as follows. We define the *weighted degree* of each variable t_i by $\text{wt}(t_i) := i$ and each monomial by $\text{wt}(t_{i_1} \cdots t_{i_N}) := \sum_{j=1}^N \text{wt}(t_{i_j})$. Then A is weighted homogeneous of weighted degree k if all monomial terms in A have weighted degree k .

Definition 3.2. A polynomial of degree $2m$, $m \geq 1$, in one variable with complex coefficients is an *ECO polynomial* if it can be written as the square of a polynomial of degree m .

Proposition 3.3. For any positive integer m , there exists a weighted homogeneous polynomial $A_k(t_1, \dots, t_m)$ of weighted degree k for each k , $m+1 \leq k \leq 2m$, such that a polynomial in one variable λ of degree $2m$

$$a_{2m}\lambda^{2m} + a_{2m-1}\lambda^{2m-1} + \cdots + a_1\lambda + 1$$

is an ECO polynomial if and only if $a_k = A_k(a_1, \dots, a_m)$ for each $m+1 \leq k \leq 2m$.

Remark 3.4. Our proof below gives a recursive formula for A_k , but an explicit expression of the polynomials A_k will not be needed in this paper.

Proof. Suppose that

$$a_{2m}\lambda^{2m} + a_{2m-1}\lambda^{2m-1} + \cdots + a_1\lambda + 1$$

is an ECO polynomial. We can find $(\sigma_1, \dots, \sigma_m) \in \mathbb{C}^m$ such that

$$a_{2m}\lambda^{2m} + a_{2m-1}\lambda^{2m-1} + \cdots + a_1\lambda + 1 = (\sigma_m\lambda^m + \sigma_{m-1}\lambda^{m-1} + \cdots + \sigma_1\lambda + 1)^2.$$

For convenience, define $\sigma_0 = 1, \sigma_{m+1} = \cdots = \sigma_{2m} = 0$, so that we can write, for each $k, 1 \leq k \leq 2m$,

$$a_k = \sum_{i=0}^k \sigma_i \sigma_{k-i}.$$

Using

$$a_k = \sum_{i=0}^k \sigma_i \sigma_{k-i} = \sum_{i=1}^{k-1} \sigma_i \sigma_{k-i} + 2\sigma_k,$$

we have

$$\sigma_k = \frac{a_k - \sum_{i=1}^{k-1} \sigma_i \sigma_{k-i}}{2}$$

for $k = 1, 2, \dots, m$. Thus

$$\sigma_1 = \frac{a_1}{2}, \quad \sigma_2 = \frac{a_2 - \sigma_1^2}{2} = \frac{a_2}{2} - \frac{a_1^2}{8}, \quad \dots$$

Using induction on k , we see that

$$\sigma_k = G_k(a_1, \dots, a_m) \text{ for each } k, 1 \leq k \leq m,$$

where $G_k(t_1, \dots, t_m)$ is a weighted homogeneous polynomial of weighted degree k . Setting $G_0 = 1, G_{m+1} = \cdots = G_{2m} = 0$, we see that

$$a_k = \sum_{\ell=0}^k G_\ell(a_1, \dots, a_m) G_{k-\ell}(a_1, \dots, a_m)$$

for all $m+1 \leq k \leq 2m$. Define

$$A_k(t_1, \dots, t_m) := \sum_{\ell=0}^k G_\ell(t_1, \dots, t_m) G_{k-\ell}(t_1, \dots, t_m).$$

Then A_k is a weighted homogeneous polynomial of weighted degree k such that $a_k = A_k(a_1, \dots, a_m)$ for each $m+1 \leq k \leq 2m$.

Conversely, given any $(a_1, \dots, a_m) \in \mathbb{C}^m$, let

$$a_{m+i} = A_{m+i}(a_1, \dots, a_m) \text{ for each } 1 \leq i \leq m$$

where A_{m+i} is defined above. Then for $\sigma_i = G_j(a_1, \dots, a_m)$, we see that

$$(\sigma_m \lambda^m + \dots + \sigma_1 \lambda + 1)^2 = a_{2m} \lambda^{2m} + a_{2m-1} \lambda^{2m-1} + \dots + a_1 \lambda + 1$$

and

$$a_{2m} \lambda^{2m} + a_{2m-1} \lambda^{2m-1} + \dots + a_1 \lambda + 1$$

is an ECO polynomial. \square

Corollary 3.5. *Regard the affine space*

$$\mathbb{A}^{2m} := \{(a_{2m}, a_{2m-1}, \dots, a_1) \mid a_i \in \mathbb{C}\}$$

as the set of polynomials

$$a_{2m} \lambda^{2m} + a_{2m-1} \lambda^{2m-1} + \dots + a_1 \lambda + 1$$

of degree $2m$ with the constant term 1. Then the set $\mathcal{D} \subset \mathbb{A}^{2m}$ of ECO-polynomials is a smooth complete intersection of m divisors D_1, \dots, D_m where D_j is the smooth divisor defined by $a_{m+j} = A_{m+j}(a_1, \dots, a_m)$ where A_{m+j} is the weighted homogeneous polynomial of weighted degree $m+j$ defined in Proposition 3.3.

Using Corollary 3.5, we will study the space of ECO lines defined in Definition 2.3. For our computation, we introduce the following notation.

Notation 3.6. Choose a homogeneous coordinate system t_0, \dots, t_n on \mathbb{P}^n . Let $\mathbb{P}_\infty^{n-1} \subset \mathbb{P}^n$ be the hyperplane defined by $t_0 = 0$. The restriction of t_1, \dots, t_n on \mathbb{P}_∞^{n-1} will be denoted by z_1, \dots, z_n . They provide a homogeneous coordinate system on \mathbb{P}_∞^{n-1} . Define the projective isomorphism $v_y : \mathbb{P}_\infty^{n-1} \rightarrow \mathbb{P}T_y(\mathbb{P}^n)$ at each point $y = [1 : y_1 : \dots : y_n] \in \mathbb{P}^n \setminus \mathbb{P}_\infty^{n-1}$ by sending $[z_1 : \dots : z_n] \in \mathbb{P}_\infty^{n-1}$ to the tangent direction of the line

$$\{(y_1 + \lambda z_1, \dots, y_n + \lambda z_n) \mid \lambda \in \mathbb{C}\}$$

at the point y . The collection $\{v_y^{-1} \mid y \in \mathbb{P}^n \setminus \mathbb{P}_\infty^{n-1}\}$ determines a canonical trivialization of the projectivized tangent bundle

$$v^{-1} : \mathbb{P}T(\mathbb{P}^n \setminus \mathbb{P}_\infty^{n-1}) \cong (\mathbb{P}^n \setminus \mathbb{P}_\infty^{n-1}) \times \mathbb{P}_\infty^{n-1}.$$

Definition 3.7. For a homogeneous polynomial $f(t_0, \dots, t_n)$ of degree $2m$, $2 \leq m \leq n-1$ and for each integer k , $0 \leq k \leq 2m$, define $a_k^f(y; z) = a_k^f(y_1, \dots, y_n; z_1, \dots, z_n)$ to be the polynomial in $2n$ variables satisfying

$$f(1, y_1 + \lambda z_1, \dots, y_n + \lambda z_n) = a_0^f(y; z) + a_1^f(y; z)\lambda + \dots + a_{2m}^f(y; z)\lambda^{2m}.$$

Note that for a fixed y , $a_k^f(y; z)$ is a homogeneous polynomial in z of degree k . In particular, $a_0^f(y; z) = f(1, y_1, \dots, y_n)$ is independent of z .

Proposition 3.8. *In Notation 3.6 and Definition 3.7, let $Y \subset \mathbb{P}^n$ be the hypersurface defined by $f(t_0, \dots, t_n) = 0$. For any point $y \in \mathbb{P}^n \setminus (Y \cup \mathbb{P}_\infty^{n-1})$, the variety $v_y^{-1}(\mathcal{E}_y^Y) \subset \mathbb{P}_\infty^{n-1}$ is (set-theoretically) the common zero set of the homogeneous polynomials in z , $B_k^f(y; z)$, $m+1 \leq k \leq 2m$, defined by*

$$B_k^f(y; z) = B_k^f(y_1, \dots, y_n; z_1, \dots, z_n) := \frac{a_k^f(y; z)}{f(1, y_1, \dots, y_n)} - A_k \left(\frac{a_1^f(y; z)}{f(1, y_1, \dots, y_n)}, \dots, \frac{a_m^f(y; z)}{f(1, y_1, \dots, y_n)} \right),$$

where A_k is as in Proposition 3.3. Note that $B_k^f(y; z)$ is homogeneous in z of degree k because $a_k^f(y; z)$ is homogeneous in z of degree k and A_k is weighted homogeneous of weighted degree k . In particular, if \mathcal{E}_y^Y is of pure dimension $n - m - 1$, then it is set-theoretically a complete intersection of multi-degree $(m + 1, m + 2, \dots, 2m)$.

Proof. A point $[z_1 : \dots : z_n] \in \mathbb{P}_\infty^{n-1}$ belongs to $v_y^{-1}(\mathcal{E}_y^Y)$ if and only if the polynomial in λ

$$f(1, y_1 + \lambda z_1, \dots, y_n + \lambda z_n) = a_0^f(y; z) + a_1^f(y; z)\lambda + \dots + a_{2m}^f(y; z)\lambda^{2m}$$

is an ECO polynomial. By Corollary 3.5, we see that $v_y^{-1}(\mathcal{E}_y^Y)$ is the common zero set of $B_k^f(y; z)$, $m + 1 \leq k \leq 2m$. \square

Proposition 3.9. *Given a general smooth complete intersection $Z \subset \mathbb{P}^{n-1}$ of multi-degree $(m + 1, \dots, 2m)$, there exist a smooth hypersurface $Y \subset \mathbb{P}^n$ of degree $2m$ and a point $y \in \mathbb{P}^n \setminus Y$, such that $\mathcal{E}_y^Y \subset \mathbb{P}T_y(\mathbb{P}^n)$ is projectively equivalent to $Z \subset \mathbb{P}^{n-1}$. In particular, for a general hypersurface $Y \subset \mathbb{P}^n$ of degree $2m$ and a general $y \in \mathbb{P}^n \setminus Y$, the variety of ECO lines $\mathcal{E}_y^Y \subset \mathbb{P}T_y(\mathbb{P}^n)$ is a smooth complete intersection of degree $(m + 1, \dots, 2m)$.*

Proof. Denote by $\{b_k(z_1, \dots, z_n) \mid m + 1 \leq k \leq 2m\}$ homogeneous polynomials with $\deg b_k = k$ defining Z . By the generality of Z , we may assume that

(i) the affine hypersurface

$$1 + b_{m+1}(t_1, \dots, t_n) + \dots + b_{2m}(t_1, \dots, t_n) = 0$$

in $\mathbb{C}^n = \{(t_1, \dots, t_n), t_i \in \mathbb{C}\}$ is smooth and

(ii) the projective hypersurface

$$b_{2m}(z_1, \dots, z_n) = 0$$

in \mathbb{P}^{n-1} with homogeneous coordinates $(z_1 : z_2 : \dots : z_n)$ is smooth.

Let $Y \subset \mathbb{P}^n$ be the hypersurface of degree $2m$ defined by the polynomial

$$f(t_0, t_1, \dots, t_n) := t_0^{2m} + t_0^{m-1}b_{m+1}(t_1, \dots, t_n) + \dots + t_0 b_{2m-1}(t_1, \dots, t_n) + b_{2m}(t_1, \dots, t_n).$$

Then Y is a smooth hypersurface because it has no singular point on its intersection with the hyperplane $t_0 = 0$ by the assumption (ii), while it has no singular point on the affine space $t_0 \neq 0$ by the assumption (i). In Notation 3.6, consider the point $y = [1 : 0 : 0 : \dots : 0] \in \mathbb{P}^n \setminus (\mathbb{P}_\infty^{n-1} \cup Y)$. Then

$$f(1, y_1 + \lambda z_1, \dots, y_n + \lambda z_n) = f(1, \lambda z_1, \dots, \lambda z_n) = 1 + \lambda^{m+1}b_{m+1}(z) + \dots + \lambda^{2m}b_{2m}(z).$$

Comparing with Definition 3.7, we obtain

$$a_0^f(y; z) = 1, a_1^f(y; z) = \dots = a_m^f(y; z) = 0, a_k^f(y; z) = b_k(z) \text{ for } m + 1 \leq k \leq 2m.$$

In the notation of Proposition 3.8,

$$B_k^f(y; z_1, \dots, z_n) = a_k^f(z_1, \dots, z_n) = b_k(z_1, \dots, z_n)$$

for $m + 1 \leq k \leq 2m$. This implies that $\mathcal{E}_y^Y, y := [1 : 0 : \dots : 0]$, is projectively equivalent to $Z \subset \mathbb{P}^{n-1}$. \square

Proof of Theorem 1.1 and Theorem 1.2. Theorem 1.2 is a direct consequence of Proposition 2.6 and Proposition 3.9.

To prove Theorem 1.1, it suffices to show by Proposition 2.6 that for any smooth $Y \subset \mathbb{P}^n$ and a general point $x \in \mathbb{P}^n \setminus Y$, the subvariety $\mathcal{E}_x^Y \subset \mathbb{P}T_x(\mathbb{P}^n)$ is a smooth complete intersection of multi-degree $(m + 1, \dots, 2m)$. Proposition 3.9 says that this is O.K. if Y is

To check it for any smooth $Y \subset \mathbb{P}^n$, choose a deformation $\{Y_t \mid |t| < \epsilon\}$ of $Y = Y_0$ such that for a Zariski open subset $U_t \subset \mathbb{P}^n \setminus Y_t$, $\mathcal{E}_x^{Y_t} \subset \mathbb{P}T_x(\mathbb{P}^n)$ is a smooth complete intersection for any $t \neq 0$ and any $x \in U_t$. By shrinking ϵ if necessary, the intersection $\cap_{t \neq 0} U_t$ is nonempty. By Proposition 2.6, we have a Zariski open subset $U \subset \mathbb{P}^n \setminus Y$ such that \mathcal{E}_x^Y is smooth for any $x \in U$. Pick a point $x \in (\cap_{t \neq 0} U_t) \cap U$.

We can construct a smooth family $\{\phi_t : X_t \rightarrow \mathbb{P}^n \mid |t| < \epsilon\}$ of double covers of \mathbb{P}^n branched along Y_t 's. Choose $z_t \in \phi_t^{-1}(x)$ in a continuous way. The proof (e.g. II.3.11.5 in [8]) of the smoothness of \mathcal{K}_x mentioned in Definition 2.2 works for the family \mathcal{K}_{z_t} , i.e., the family $\{\mathcal{K}_{z_t} \mid |t| < \epsilon\}$ is a flat family of nonsingular projective subvarieties. Via Proposition 2.6, this implies that $\{\mathcal{E}_x^{Y_t} \mid |t| < \epsilon\}$ is a flat family of nonsingular projective varieties in $\mathbb{P}T_x(\mathbb{P}^n)$. By our choice of x , $\mathcal{E}_x^{Y_t}$ is (scheme-theoretically) a smooth complete intersection for $t \neq 0$, while $\mathcal{E}_x^{Y_0}$ is a nonsingular variety which is set-theoretically a complete intersection of the same multi-degree as $\mathcal{E}_x^{Y_t}$, $t \neq 0$, by Proposition 3.8. We conclude that $\mathcal{E}_x^{Y_0}$ is also a smooth complete intersection of multi-degree $(m+1, \dots, 2m)$. \square

4. VARIATION OF VMRT

Notation 4.1. Let V_k be the vector space of homogeneous polynomials of degree k in z_1, \dots, z_n . Each polynomial $h \in V_k$ is of the form

$$h(z_1, \dots, z_n) = \sum_{i_1 + \dots + i_n = k} e_{i_1, \dots, i_n} z_1^{i_1} \cdots z_n^{i_n}.$$

Regarding V_k as a complex manifold, take $\{e_{i_1, \dots, i_n}\}_{i_1 + \dots + i_n = k}$ as linear coordinates on V_k and

$$\left\{ \frac{\partial}{\partial e_{i_1, \dots, i_n}} \mid i_1 + \dots + i_n = k \right\}$$

as a basis for the tangent spaces $T_h(V_k)$ of V_k at each $h \in V_k$. There is a canonical isomorphism between V_k and $T_h(V_k)$ identifying a polynomial

$$\sum_{i_1 + \dots + i_n = k} E_{i_1, \dots, i_n} z_1^{i_1} \cdots z_n^{i_n} \in V_k$$

with the tangent vector

$$\sum_{i_1 + \dots + i_n = k} E_{i_1, \dots, i_n} \frac{\partial}{\partial e_{i_1, \dots, i_n}} \in T_h(V_k).$$

Notation 4.2. For a homogeneous polynomial $f(t_0, \dots, t_n)$ of degree $2m$, $2 \leq m \leq n-1$, write

$$f(t_0, \dots, t_n) = t_0^{2m} f_0(t_1, \dots, t_n) + t_0^{2m-1} f_1(t_1, \dots, t_n) + \cdots + f_{2m}(t_1, \dots, t_n)$$

where $f_k(t_1, \dots, t_n)$ is a homogeneous polynomial of degree $k = 0, \dots, 2m$ in t_1, \dots, t_n . Comparing with Definition 3.7, we have

$$f(1, 0, \dots, 0) = f_0(z_1, \dots, z_n) = a_0^f(0; z) \text{ and } a_i^f(0; z) = f_i(z) \text{ for } i = 1, \dots, 2m.$$

Definition 4.3. For a homogeneous polynomial $f(t_0, \dots, t_n)$ of degree $2m$, $2 \leq m \leq n-1$, let $Y \subset \mathbb{P}^n$ be the hypersurface defined by $f(t_0, \dots, t_n) = 0$ and define a morphism

$$\mu : \mathbb{P}^n \setminus (\mathbb{P}_\infty^{n-1} \cup Y) \rightarrow V_{m+1}$$

by sending $y = [1 : y_1 : \dots : y_n]$ to the polynomial in z

$$\mu(y) := [B_{m+1}^f(y; z)] \in V_{m+1}$$

with $B_{m+1}^f(y; z)$ as in Proposition 3.8.

Proposition 4.4. *In Notation 4.2 and Definition 4.3, assume that*

$$f(1, 0, \dots, 0) = f_0(z) = 1 \text{ and } f_1(z) = \dots = f_m(z) = 0.$$

Then for $x = [1 : 0 : \dots : 0] \in \mathbb{P}^n \setminus (\mathbb{P}_\infty^{n-1} \cup Y)$ and $\sum_{i=1}^n v_i \frac{\partial}{\partial y_i} \in T_x(\mathbb{P}^n)$,

$$d\mu_x\left(\sum_{i=1}^n v_i \frac{\partial}{\partial y_i}\right) = \left[\sum_{i=1}^n v_i \frac{\partial f_{m+2}}{\partial t_i}(z)\right] \in T_{\mu(x)}(V_{m+1}) = V_{m+1}.$$

We will use the following lemma.

Lemma 4.5. *In Notation 4.2, set $f_{2m+1} = 0$ for convenience. Assume that*

$$f(1, 0, \dots, 0) = f_0(z) = 1.$$

Then $B_k^f(y; z)$ of Proposition 3.8 satisfies

$$\begin{aligned} \left. \frac{\partial B_k^f(y; z)}{\partial y_i} \right|_{(0; z)} &= \frac{\partial f_{k+1}}{\partial t_i}(z) - f_k(z) \frac{\partial f_1}{\partial t_i}(z) \\ &\quad - \sum_{j=1}^m \frac{\partial A_k}{\partial x_j}(f_1(z), \dots, f_m(z)) \left(\frac{\partial f_{j+1}}{\partial t_i}(z) - f_j(z) \frac{\partial f_1}{\partial t_i}(z) \right) \end{aligned}$$

for all $k = m + 1, \dots, 2m$ and $i = 1, \dots, n$.

Proof. In the equality

$$\left. \frac{\partial f(t)}{\partial t_i} \right|_{t=(1, \lambda z_1, \dots, \lambda z_n)} = \left. \frac{\partial f(1, y_1 + \lambda z_1, \dots, y_n + \lambda z_n)}{\partial y_i} \right|_{y_1 = \dots = y_n = 0},$$

the left hand side can be written, via Notation 4.2,

$$\frac{\partial f_1}{\partial t_i}(z_1, \dots, z_n) + \frac{\partial f_2}{\partial t_i}(z_1, \dots, z_n)\lambda + \dots + \frac{\partial f_{2m+1}}{\partial t_i}(z_1, \dots, z_n)\lambda^{2m}.$$

On the other hand, the right hand side is, by Definition 3.7,

$$\frac{\partial a_0^f}{\partial y_i}(0; z) + \frac{\partial a_1^f}{\partial y_i}(0; z)\lambda + \dots + \frac{\partial a_{2m}^f}{\partial y_i}(0; z)\lambda^{2m}.$$

Therefore for each $i = 1, \dots, n$,

$$\frac{\partial a_{2m}^f}{\partial y_i}(0; z) = 0 \text{ and } \frac{\partial a_k^f}{\partial y_i}(0; z) = \frac{\partial f_{k+1}}{\partial t_i}(z) \text{ for } k = 0, \dots, 2m.$$

From this and the assumption that $a_0^f(0; z) = f(1, 0, \dots, 0) = 1$, we obtain

$$\begin{aligned} \left. \frac{\partial}{\partial y_i} \left(\frac{a_k^f(y; z)}{a_0^f(y; z)} \right) \right|_{(y; z) = (0; z)} &= \frac{a_0^f(0; z) \frac{\partial a_k^f}{\partial y_i}(0; z) - a_k^f(0; z) \frac{\partial a_0^f}{\partial y_i}(0; z)}{a_0^f(0; z)^2} \\ &= \frac{\partial f_{k+1}}{\partial t_i}(z) - f_k(z) \frac{\partial f_1}{\partial t_i}(z) \end{aligned}$$

for all $k = 0, \dots, 2m$ and $i = 1, \dots, n$. Thus

$$\begin{aligned} \frac{\partial B_k^f(y; z)}{\partial y_i} \Big|_{(0; z)} &= \frac{\partial}{\partial y_i} \left(\frac{a_k^f(y; z)}{a_0^f(y; z)} \right) \Big|_{(0; z)} \\ &\quad - \sum_{j=1}^m \frac{\partial A_k}{\partial x_j} \left(\frac{a_1^f(0; z)}{a_0^f(0; z)}, \dots, \frac{a_m^f(0; z)}{a_0^f(0; z)} \right) \frac{\partial}{\partial y_i} \left(\frac{a_j^f(y; z)}{a_0^f(y; z)} \right) \Big|_{(0; z)} \\ &= \frac{\partial f_{k+1}}{\partial t_i}(z) - f_k(z) \frac{\partial f_1}{\partial t_i}(z) \\ &\quad - \sum_{j=1}^m \frac{\partial A_k}{\partial x_j}(f_1(z), \dots, f_m(z)) \left(\frac{\partial f_{j+1}}{\partial t_i}(z) - f_j(z) \frac{\partial f_1}{\partial t_i}(z) \right) \end{aligned}$$

for all $k = m + 1, \dots, 2m$ and $i = 1, \dots, n$. \square

Proof of Proposition 4.4. Since $A_k(x_1, \dots, x_m)$ is weighted homogeneous of weighted degree k , if $k \geq m + 1$, then the linear part of $A_k(x_1, \dots, x_m)$ does not contain variables x_1, \dots, x_m . Therefore for all $k = m + 1, \dots, 2m$ and $j = 1, \dots, m$,

$$\frac{\partial A_k}{\partial x_j} \Big|_{(0, \dots, 0)} = 0.$$

Thus putting $f_1 = \dots = f_m = 0$ in Lemma 4.5, we obtain

$$\frac{\partial B_{m+1}^f}{\partial y_i}(0; z) = \frac{\partial f_{m+2}}{\partial t_i}(z).$$

It follows that

$$d\mu_x \left(\sum_{i=1}^n v_i \frac{\partial}{\partial y_i} \right) = \sum_{i=1}^n v_i \frac{\partial B_{m+1}^f}{\partial y_i}(0; z) = \sum_{i=1}^n v_i \frac{\partial f_{m+2}}{\partial t_i}(z).$$

\square

Notation 4.6. Denote the action of $A \in GL(n, \mathbb{C})$ on \mathbb{C}^n by $(z_1, \dots, z_n) \mapsto A(z_1, \dots, z_n)$. We have the natural induced action on V_k given by

$$(A.h)(z_1, \dots, z_n) := h(A^{-1}(z_1, \dots, z_n)), \quad h \in V_k.$$

Denote the orbit of $h \in V_k$ by

$$GL(n, \mathbb{C}).h := \{A.h \mid A \in GL(n, \mathbb{C})\} \subset V_k.$$

Proposition 4.7. *We use the terminology of Notation 4.1 and Notation 4.6. A tangent vector*

$$\sum_{i_1 + \dots + i_n = k} E_{i_1, \dots, i_n} \frac{\partial}{\partial e_{i_1, \dots, i_n}} \in T_h(V_k)$$

is tangent to the orbit $GL(n, \mathbb{C}).h$ if and only if there exists an $(n \times n)$ matrix $(s_j^i)_{i,j=1, \dots, n}$ such that

$$\sum_{i_1 + \dots + i_n = k} E_{i_1, \dots, i_n} z_1^{i_1} \cdots z_n^{i_n} = \frac{d}{dt} h \left(z_1 + t \sum_{i=1}^n s_1^i z_i, \dots, z_n + t \sum_{i=1}^n s_n^i z_i \right) \Big|_{t=0}.$$

Proof. Define a morphism

$$GL(n, \mathbb{C}) \times V_k \rightarrow V_k$$

sending A to $A.h$. Then $GL(n, \mathbb{C}).h$ is the image of α_h and $\alpha_h(I) = h$ where I is the $(n \times n)$ identity matrix. The tangent space of $GL(n, \mathbb{C}).h$ at h is the image of the differential

$$d(\alpha_h)_I : T_I(GL(n, \mathbb{C})) \rightarrow T_h(V_k).$$

Let us identify $T_I(GL(n, \mathbb{C}))$ with the vector space M_n of all $(n \times n)$ matrices so that $A \in M_n$ corresponds to the tangent vector at I of the curve $c(t) = I + tA$ which is indeed a curve on $GL(n, \mathbb{C})$ for sufficiently small t . Since $\alpha_h \circ c(t)$ is the polynomial $h((I + tA)^{-1}(z_1, \dots, z_n))$, the differential $d(\alpha_h)_I$ sends $A = \frac{d}{dt}c(t)\big|_{t=0}$ to $\frac{d}{dt}h((I + tA)^{-1}(z_1, \dots, z_n))\big|_{t=0}$ which is of the form on the right hand side of the equation in the proposition. \square

Proposition 4.8. *In the setting of Proposition 4.4,*

$$T_{\mu(x)}(GL(n, \mathbb{C}).\mu(x)) = \left\{ \sum_{i,j=1}^n s_j^i z_i \frac{\partial f_{m+1}}{\partial t_j}(z) \mid s_j^i \in \mathbb{C} \right\} \subset T_{\mu(x)}(V_{m+1}) = V_{m+1}.$$

Proof. From $\mu(x) = [B_{m+1}(0; z)] \in V_{m+1}$ and Proposition 4.7, we get

$$T_{\mu(x)}(GL(n, \mathbb{C}).\mu(x)) = \left\{ \frac{d}{dt} B_{m+1}^f(0; z_1 + t \sum_{i=1}^n s_1^i z_i, \dots, z_n + t \sum_{i=1}^n s_n^i z_i) \Big|_{t=0} \mid s_j^i \in \mathbb{C} \right\}.$$

From Notation 4.2, we have $a_0^f(0; z) = f(1, 0, \dots, 0) = 1$ and $a_i^f(0; z) = f_i(z) = 0$ for $i = 1, \dots, m$. Thus

$$B_{m+1}^f(0; z) = \frac{a_{m+1}^f(0; z)}{a_0^f(0; z)} - A_{m+1} \left(\frac{a_1^f(0; z)}{a_0^f(0; z)}, \dots, \frac{a_m^f(0; z)}{a_0^f(0; z)} \right) = f_{m+1}(z).$$

So the following equalities hold

$$\begin{aligned} & \frac{d}{dt} B_{m+1}^f(0; z_1 + t \sum_{i=1}^n s_1^i z_i, \dots, z_n + t \sum_{i=1}^n s_n^i z_i) \Big|_{t=0} \\ &= \frac{d}{dt} f_{m+1}(z_1 + t \sum_{i=1}^n s_1^i z_i, \dots, z_n + t \sum_{i=1}^n s_n^i z_i) \Big|_{t=0} \\ &= \sum_{i,j=1}^n s_j^i z_i \frac{\partial f_{m+1}}{\partial t_j}(z). \end{aligned}$$

Putting it in the above expression for $T_{\mu(x)}(GL(n, \mathbb{C}).\mu(x))$, we obtain the result. \square

Proposition 4.9. *There exists a smooth hypersurface $Y \subset \mathbb{P}^n$, $n \geq 4$, defined by a homogeneous polynomial f of degree $2m$, $2 \leq m \leq n-1$, such that, for a general $x \in \mathbb{P}^n \setminus (\mathbb{P}_\infty^{n-1} \cup Y)$, using the terminology of Definition 4.3,*

$$\text{rank}(d\mu_x) = n, \dim_{\mathbb{C}} T_{\mu(x)}(GL(n, \mathbb{C}).\mu(x)) = n^2 \text{ and}$$

$$\text{Im}(d\mu_x) \cap T_{\mu(x)}(GL(n, \mathbb{C}).\mu(x)) = 0.$$

Proof. First, consider the case $m = 2$. Set

$$f(t_1, \dots, t_n) = t_0^4 + b(t_1^3 + \dots + t_n^3)t_0 + (t_1^4 + \dots + t_n^4) + \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n} ct_{i_1}t_{i_2}t_{i_3}t_{i_4}$$

with some constants $b, c \in \mathbb{C}^*$. Using Notation 4.2, we have

$$f_4 = (t_1^4 + \cdots + t_n^4) + \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n} ct_{i_1} t_{i_2} t_{i_3} t_{i_4}.$$

Since the Fermat hypersurface in \mathbb{P}^n defined by $t_0^4 + t_1^4 + \cdots + t_n^4 = 0$ is smooth, the hypersurface Y defined by $f = 0$ is smooth if we choose general b and c . Set $x := [1 : 0 : \cdots : 0]$. By Propositions 4.4 and 4.8, we have

$$\text{Im}(d\mu_x) = \left\{ \sum_{i=1}^n v_i \frac{\partial f_4}{\partial t_i}(z) \mid v_i \in \mathbb{C} \right\} = \left\{ \sum_{i=1}^n v_i (4z_i^3 + \sum_{1 \leq i_1 < i_2 < i_3 \leq n, \forall i_k \neq i} cz_{i_1} z_{i_2} z_{i_3}) \mid v_i \in \mathbb{C} \right\}$$

and

$$T_{\mu(x)}(GL(n, \mathbb{C}).\mu(x)) = \left\{ \sum_{i,j=1}^n s_j^i z_i \frac{\partial f_3}{\partial t_j}(z) \mid s_j^i \in \mathbb{C} \right\} = \left\{ \sum_{i,j=1}^n s_j^i z_i z_j^2 \mid s_j^i \in \mathbb{C} \right\}.$$

From this it follows that $\text{rank}(d\mu_x) = n$ and $\dim_{\mathbb{C}} T_{\mu(x)}(GL(n, \mathbb{C}).\mu(x)) = n^2$. Also if there exist s_j^i and v_i such that

$$\sum_{i,j=1}^n s_j^i z_i z_j^2 = \sum_{i=1}^n v_i (4z_i^3 + \sum_{1 \leq i_1 < i_2 < i_3 \leq n, \forall i_k \neq i} cz_{i_1} z_{i_2} z_{i_3}),$$

then $s_j^i = 0$ and $v_i = 0$ for all i and j . Therefore

$$\text{Im}(d\mu_x) \cap T_{\mu(x)}(GL(n, \mathbb{C}).\mu(x)) = 0.$$

Next, assume that $m \geq 3$. Pick

$$f(t_0, \dots, t_n) = t_0^{2m} + b(t_1^{m+1} + \cdots + t_n^{m+1})t_0^{m-1} + ct_1 t_2 t_3 (t_4^{m-1} + \cdots + t_n^{m-1})t_0^{m-2} + t_1^{2m} + \cdots + t_n^{2m}$$

with some constants $b, c \in \mathbb{C}^*$. Using Notation 4.2, we have

$$f_1 = \cdots = f_m = f_{m+3} = \cdots = f_{2m-1} = 0, \quad f_{m+1} = b(t_1^{m+1} + \cdots + t_n^{m+1}),$$

$$f_{m+2} = ct_1 t_2 t_3 (t_4^{m-1} + \cdots + t_n^{m-1}) \text{ and } f_{2m} = t_1^{2m} + \cdots + t_n^{2m}.$$

From the smoothness of the Fermat hypersurface in \mathbb{P}^n defined by $t_0^{2m} + t_1^{2m} + \cdots + t_n^{2m} = 0$, we can see that the hypersurface Y defined by $f = 0$ is smooth for general b and c . Set $x := [1 : 0 : \cdots : 0]$. Propositions 4.4 and 4.8 show that

$$\begin{aligned} \text{Im}(d\mu_x) &= \left\{ \sum_{i=1}^n v_i \frac{\partial f_{m+2}}{\partial t_i}(z) \mid v_i \in \mathbb{C} \right\} \\ &= \left\{ (v_1 z_2 z_3 + v_2 z_1 z_3 + v_3 z_1 z_2)(z_4^{m-1} + \cdots + z_n^{m-1}) + \sum_{i=4}^n v_i z_1 z_2 z_3 z_i^{m-2} \mid v_i \in \mathbb{C} \right\} \end{aligned}$$

and

$$T_{\mu(x)}(GL(n, \mathbb{C}).\mu(x)) = \left\{ \sum_{i,j=1}^n s_j^i z_i \frac{\partial f_{m+1}}{\partial t_j}(z) \mid s_j^i \in \mathbb{C} \right\} = \left\{ \sum_{i,j=1}^n s_j^i z_i z_j^m \mid s_j^i \in \mathbb{C} \right\}.$$

The condition $m \geq 3$ implies that $\text{rank}(d\mu_x) = n$. It is easy to see that

$$\dim_{\mathbb{C}} T_{\mu(x)}(GL(n, \mathbb{C}).\mu(x)) = n^2 \text{ and } \text{Im}(d\mu_x) \cap T_{\mu(x)}(GL(n, \mathbb{C}).\mu(x)) = 0.$$

Proof of Theorem 1.3. By Proposition 2.6, we may prove the corresponding statement for the morphism $\eta : W \rightarrow \text{Hilb}(\mathbb{P}^{n-1})$ defined on a neighborhood W of a general point in \mathbb{P}^n by $\eta(y) := [\mathcal{E}_y^Y]$ for $y \in W$. Since \mathcal{E}_x^Y is a complete intersection of multi-degree $(m+1, \dots, 2m)$ for general Y and general $x \in \mathbb{P}^n \setminus Y$, the equation B_{m+1} of degree $m+1$ is uniquely determined up to $GL(n, \mathbb{C})$ -action by the projective equivalence class of \mathcal{E}_x^Y . Thus it suffices to show that $\text{rank}(d\mu_x) = n$ and $d\mu_x(T_x(\mathbb{P}^n)) \cap T_{\mu(x)}(GL(n, \mathbb{C}) \cdot \mu(x)) = 0$ for a general Y and general x . This follows from Proposition 4.9. \square

5. PROJECTIVE CONNECTIONS AND RIGIDITY OF MAPS

Definition 5.1. Given a complex manifold M of dimension n , the projectivized tangent bundle $\pi : \mathbb{P}T(M) \rightarrow M$ is equipped with the tautological line bundle $\xi \subset \pi^*T(M)$ whose fiber at $\alpha \in \mathbb{P}T(M)$ is given by $\hat{\alpha} \subset T_{\pi(\alpha)}(M)$, the 1-dimensional subspace corresponding to $\alpha \in \mathbb{P}T(M)$. We have the vector subbundle $\mathcal{T} \subset T(\mathbb{P}T(M))$ of rank n whose fiber at $\alpha \in \mathbb{P}T(M)$ is given by

$$\mathcal{T}_\alpha := d\pi_\alpha^{-1}(\hat{\alpha})$$

where $d\pi_\alpha : T_\alpha(\mathbb{P}T(M)) \rightarrow T_{\pi(\alpha)}(M)$ is the differential of the projection π . A *projective connection* on M is a homomorphism $p : \xi \rightarrow \mathcal{T}$ of vector bundles which splits the exact sequence of vector bundles on $\mathbb{P}T(M)$

$$0 \longrightarrow T^\pi \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/T^\pi \cong \xi \longrightarrow 0$$

where $T^\pi \subset T(\mathbb{P}T(M))$ is the relative tangent bundle of π . Given a projective connection $p : \xi \rightarrow \mathcal{T}$, the image $p(\xi) \subset \mathcal{T} \subset T(\mathbb{P}T(M))$ is a line subbundle in the tangent bundle of $\mathbb{P}T(M)$ and defines a foliation of rank 1 on $\mathbb{P}T(M)$.

Example 5.2. On \mathbb{P}^n , we have a canonical projective connection $p : \xi \rightarrow \mathcal{T}$ such that the leaves of the foliation $p(\xi)$ are exactly the tangent directions of lines on \mathbb{P}^n . We call this the *flat projective connection* and denote it by p^{flat} . Let U be a connected complex manifold of dimension n and let $\varphi : U \rightarrow \mathbb{P}^n$ be an immersion. Via the biholomorphic morphism $\mathbb{P}T(U) \cong \mathbb{P}T(\varphi(U))$, we have an induced projective connection φ^*p^{flat} on U . By the affirmative answer to Problem 1.8 when $X = \mathbb{P}^n$ (see the remark after Problem 1.8), two immersions $\varphi_i : U \rightarrow \mathbb{P}^n, i = 1, 2$, are related by a projective transformation, i.e., there exists an automorphism $\psi : \mathbb{P}^n \rightarrow \mathbb{P}^n$ such that $\varphi_2 = \psi \circ \varphi_1$, if and only if the two projective connections $\varphi_1^*p^{\text{flat}}$ and $\varphi_2^*p^{\text{flat}}$ coincide.

Proposition 5.3. *In the setting of Definition 5.1, let $\mathcal{C} \subset \mathbb{P}T(M)$ be a closed subvariety dominant over M such that for a general point $x \in M$, the fiber $\mathcal{C}_x \subset \mathbb{P}T_x(M)$ is not contained in a quadric hypersurface. Suppose that $p_1, p_2 : \xi \rightarrow \mathcal{T}$ are two projective connections on M such that $p_1|_{\mathcal{C}} = p_2|_{\mathcal{C}}$. Then $p_1 = p_2$.*

Proof. Since p_1 and p_2 split the exact sequence in Definition 5.1, the difference $p_1 - p_2$ determines an element $\sigma \in H^0(\mathbb{P}T(M), T^\pi \otimes \xi^{-1})$. For a general $x \in M$, σ_x is a section of $T(\mathbb{P}T_x(M)) \otimes \xi^{-1}$ on the projective space $\mathbb{P}T_x(M)$. The condition $p_1|_{\mathcal{C}} = p_2|_{\mathcal{C}}$ implies that σ_x vanishes on the subvariety \mathcal{C}_x . In term of a homogeneous coordinate system on projective space \mathbb{P}^{n-1} , a nonzero section of $T(\mathbb{P}^{n-1}) \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ is represented by a homogeneous polynomial vector field with quadratic coefficients. In particular, the zero set of such a section must be contained in some quadric hypersurface. By the assumption that \mathcal{C}_x is not contained in a quadric hypersurface, we see that $\sigma_x = 0$. Since it is true for a general $x \in M$, we obtain $p_1 = p_2$. \square

We have the following general version of Theorem 1.4. In fact, Theorem 1.4 is a corollary

Theorem 5.4. *Let X be a Fano manifold. For a general point $x \in X$, we denote by \mathcal{K}_x the space of minimal rational curves through x and by $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ the VMRT at x . Assume that \mathcal{C}_x is not contained in a quadric hypersurface in $\mathbb{P}T_x(X)$. Let $U \subset X$ be a connected neighborhood of a general point $x \in X$ and $\varphi_1, \varphi_2 : U \rightarrow \mathbb{P}^n$ be two biholomorphic immersions such that for any $y \in U$ and any member C of \mathcal{K}_y , both $\varphi_1(C \cap U)$ and $\varphi_2(C \cap U)$ are contained in lines in \mathbb{P}^n . Then there exists a projective transformation $\psi : \mathbb{P}^n \rightarrow \mathbb{P}^n$ such that $\varphi_2 = \psi \circ \varphi_1$.*

Proof. Let $\mathcal{C} \subset \mathbb{P}T(X)$ be the closure of the union of $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ as x varies over general points of X . For a member C of \mathcal{K}_x and its smooth locus $C^\circ \subset C$, the curve $\mathbb{P}T(C^\circ) \subset \mathbb{P}T(X)$ lies in \mathcal{C} . In fact, by the definition of \mathcal{C} such curves cover a dense open subset in \mathcal{C} .

Consider the projective connections $\varphi_i^* p^{\text{flat}}$ on U . Let C be a general minimal rational curve intersecting U . Since $\varphi_1(C \cap U)$ and $\varphi_2(C \cap U)$ are contained in lines in \mathbb{P}^n , the difference

$$\varphi_1^* p^{\text{flat}} - \varphi_2^* p^{\text{flat}} \in H^0(\mathbb{P}T(U), T^\pi \otimes \xi^{-1}),$$

in the notation of Definition 5.1 with $M = U$, vanishes along the Riemann surface $\mathbb{P}T(C^\circ) \cap \mathbb{P}T(U)$. Since such Riemann surfaces cover a dense open subset in $\mathcal{C} \cap \mathbb{P}T(U)$, the two projective connections must agree on $\mathcal{C} \cap \mathbb{P}T(U)$. Applying Proposition 5.3 with $M = U$, we conclude $\varphi_1^* p^{\text{flat}} = \varphi_2^* p^{\text{flat}}$. As mentioned in Example 5.2, this implies the existence of a projective transformation ψ satisfying $\varphi_2 = \psi \circ \varphi_1$. \square

Proof of Theorem 1.7. Putting $m = n - 1$ in the proof of Proposition 2.6, we see that minimal rational curves on $X_i, i = 1, 2$, have trivial normal bundles and rational curves through general points with trivial normal bundles are minimal rational curves. By Proposition 6 of [6] (also cf. Theorem 3.1 (iv) in [11]), for a general minimal rational curve $C \subset X_2$, each irreducible component of $f^{-1}(C)$ is a minimal rational curve in X_1 . In other words, f sends minimal rational curves of X_1 through a general point to those of X_2 . Putting

$$\hat{X} = X_1, X = X_2, g = f, \phi = \phi_2, \text{ and } h = \phi_1$$

in Corollary 1.5, we see that $\phi_1 = \psi \circ \phi_2 \circ f$ for some projective transformation ψ . Thus f must be birational, and hence an isomorphism. \square

Proof of Theorem 1.9. Applying Theorem 1.4 to $\varphi := \phi_2 \circ \gamma : U_1 \rightarrow \phi_2(U_1) \subset \mathbb{P}^n$, we have a projective transformation $\psi \in \text{Aut}(\mathbb{P}^n)$ such that $\psi \circ \phi_1|_{U_1} = \phi_2 \circ \gamma$. By the assumption on γ and Proposition 2.4, we have $d\psi(\mathcal{E}_x^{Y_1}) = \mathcal{E}_{\psi(x)}^{Y_2}$ for $x \in \phi_1(U_1)$. By Proposition 2.5, this implies $\psi(Y_1) = Y_2$. Thus replacing Y_1 by $\psi(Y_1)$ and ϕ_1 by $\psi \circ \phi_1$, we may assume that $Y_1 = Y_2$ and $\phi_1(U_1) = \phi_2(U_2)$. From Lemma 2.1, there exists a biregular morphism $\Gamma : X_1 \rightarrow X_2$ with $\Gamma|_{U_1} = \gamma$. \square

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